# Integration of Rousselier's continuous ductile damage model

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Rousselier's continuous ductile damage model (Rousselier, 1981; Rousselier, 1987) is a popular alternative to GTN. The basics of integration of pressure dependent plastic model such as GTN or Rousselier's have been studied in detail (Aravas, 1987).

## 1. Preliminaries

Bold symbols denote tensors of rank 1 (vectors) and 2. Helvetica symbols denote tensors of rank 4.

Strain rate decomposition is assumed:

$$\dot{\mathbf{E}} = \dot{\boldsymbol{\varepsilon}}^e + \dot{\boldsymbol{\varepsilon}}^p$$

where $\mathbf{E}$ is the total small strain. This unusual notation for small strain helps simplify notation later. In finite differences:

$$\Delta\mathbf{E} = \Delta\boldsymbol{\varepsilon}^e + \Delta\boldsymbol{\varepsilon}^p$$

To simplify notation we introduce the following definitions:

$$\boldsymbol{\varepsilon} \equiv \Delta\boldsymbol{\varepsilon}^p \quad ; \quad \varepsilon_p \equiv \frac{1}{3}\,\mathrm{tr}\boldsymbol{\varepsilon} \quad ; \quad \mathbf{e} \equiv \boldsymbol{\varepsilon} - \varepsilon_p \mathbf{I} \quad ; \quad \varepsilon_q \equiv \left(\frac{2}{3}\,\mathbf{e}\colon\mathbf{e}\right)^{1/2}$$

$\boldsymbol{\sigma}$ is the stress tensor

$$p \equiv -\frac{1}{3}\,\mathrm{tr}\boldsymbol{\sigma} \quad ; \quad \mathbf{S} \equiv \boldsymbol{\sigma} + p\mathbf{I} \quad ; \quad q \equiv \left(\frac{3}{2}\,\mathbf{S}\colon\mathbf{S}\right)^{1/2}$$

Existence of pastic potential, $g$, is assumed. Typically, the plastic potential is the same as the flow surface, $F$. The plastic potential is a function of 2 stress invariants, $p$ and $q$, and of some internal variables, $\alpha_i$:

$$g = g(p, q, \alpha_i)$$

The normality rule is:

$$\boldsymbol{\varepsilon} = \lambda\,\frac{\partial g}{\partial\boldsymbol{\sigma}} = \lambda\left(\frac{\partial g}{\partial p}\,\frac{\partial p}{\partial\boldsymbol{\sigma}} + \frac{\partial g}{\partial q}\,\frac{\partial q}{\partial\boldsymbol{\sigma}}\right)$$

where from above

$$\frac{\partial p}{\partial\boldsymbol{\sigma}} = -\frac{1}{3}\,\mathbf{I} \quad ; \quad \frac{\partial q}{\partial\boldsymbol{\sigma}} = \frac{3\mathbf{S}}{2q} \equiv \mathbf{n}$$

so that

$$\boldsymbol{\sigma} = \frac{2}{3}\,q\mathbf{n} - p\mathbf{I} \quad ; \quad \boldsymbol{\varepsilon} = \mathbf{e} + \varepsilon_p\mathbf{I} = \lambda\left(-\frac{1}{3}\,\frac{\partial g}{\partial p}\,\mathbf{I} + \frac{\partial g}{\partial q}\,\mathbf{n}\right)$$

since $\mathbf{n}$ and $\mathbf{I}$ are orthogonal in stress-strain space, i.e. $\mathbf{n}\colon\mathbf{I} = 0$, we have

$$\mathbf{e} = \lambda\,\frac{\partial g}{\partial q}\,\mathbf{n} \quad ; \quad \varepsilon_p = -\frac{1}{3}\,\lambda\,\frac{\partial g}{\partial p}$$

and

$$\varepsilon_q \equiv (\frac{2}{3}\,\mathbf{e}\colon\mathbf{e})^{½} = \lambda\,\frac{\partial g}{\partial q}\,(\frac{2}{3}\,\mathbf{n}\colon\mathbf{n})^{½} = \lambda\,\frac{\partial g}{\partial q}\,\frac{1}{q}\,(\frac{2}{3}\,\frac{3}{2}\,\frac{3}{2}\,\mathbf{S}\colon\mathbf{S})^{½} = \lambda\,\frac{\partial g}{\partial q}$$

Expressing $\lambda$ from this and the previous expression for $\varepsilon_p$:

$$\varepsilon_q\,\frac{\partial g}{\partial p} = -3\varepsilon_p\,\frac{\partial g}{\partial q}$$

or finally the normality rule leads to:

$$\varepsilon_q\,\frac{\partial g}{\partial p} + 3\varepsilon_p\,\frac{\partial g}{\partial q} = 0$$

This important expression forms one equation in a system of non-linear equations to solve for $\varepsilon_q$ and $\varepsilon_p$. Importantly, this equation is valid for GTN and the Rousselier's model, and possibly for other pressure dependent models.

Also it's easy to see that $\boldsymbol{\varepsilon}$ can now be written as

$$\boldsymbol{\varepsilon} = \varepsilon_p\mathbf{I} + \varepsilon_q\mathbf{n}$$

which can be combined with elasticity to give the two remaining expessions. Elasticity:

$$\boldsymbol{\sigma} = \mathsf{C}\colon\boldsymbol{\varepsilon}^e$$

where $\mathsf{C}$ is rank 4 elasticity tensor and

$$\boldsymbol{\varepsilon}^e = \boldsymbol{\varepsilon}^e_t + \Delta\boldsymbol{\varepsilon}^e = \boldsymbol{\varepsilon}^e_t + \Delta\mathrm{E} - \boldsymbol{\varepsilon}$$

so that the elasticity statement becomes

$$\boldsymbol{\sigma} = \mathsf{C}\colon(\boldsymbol{\varepsilon}^e_t + \Delta\mathrm{E} - \boldsymbol{\varepsilon}) = \boldsymbol{\sigma}^e - \mathsf{C}\colon\boldsymbol{\varepsilon}$$

The elastic tensor is

$$\mathsf{C} = 2G\mathsf{I} + (K - \frac{2}{3}\,G)\mathbf{I}\otimes\mathbf{I}$$

where $\mathsf{I}$ is the rank 4 identity tensor. So

$$\mathsf{C}\colon\boldsymbol{\varepsilon} = \mathsf{C}\colon(\varepsilon_p\mathbf{I} + \varepsilon_q\mathbf{n}) = (2G\mathsf{I} + (K - \frac{2}{3}\,G)\mathbf{I}\otimes\mathbf{I})\colon(\varepsilon_p\mathbf{I} + \varepsilon_q\mathbf{n}) = 2G\varepsilon_p\mathbf{I} + 2G\varepsilon_q\mathbf{n} + 3(K - \frac{2}{3}\,G)\varepsilon_p\mathbf{I} = 2G\varepsilon_q\mathbf{n} + 3K\varepsilon_p\mathbf{I}$$

Just as with $\boldsymbol{\sigma}$, $\boldsymbol{\sigma}^e$ can be split along $\mathbf{n}$ and $\mathbf{I}$ tensors:

$$p^e = -\frac{1}{3}\,\mathrm{tr}\boldsymbol{\sigma}^e \quad ; \quad \mathbf{S}^e = \boldsymbol{\sigma}^e + p^e\mathbf{I} \quad ; \quad q^e = (\frac{3}{2}\,\mathbf{S}^e\colon\mathbf{S}^e)^{½}$$

so that

$$\boldsymbol{\sigma} = \mathbf{S} - p\mathbf{I} = \mathbf{S}^e - p^e\mathbf{I} - 2G\varepsilon_q\mathbf{n} - 3K\varepsilon_p\mathbf{I}$$

which splits along $\mathbf{n}$ and $\mathbf{I}$:

$$\mathbf{S} = \mathbf{S}^e - 2G\varepsilon_q\mathbf{n} \quad ; \quad p = p^e + 3K\varepsilon_p$$

where the first equation means that $\mathbf{S}^e \propto \mathbf{n}$, hence

$$\mathbf{n} = \frac{3\mathbf{S}^e}{2q^e} \quad \rightarrow \quad \mathbf{S}^e = \frac{2}{3}\,q^e\mathbf{n}$$

and expressing $\mathbf{S}$ via $q$:

$$\frac{2}{3}\,q\mathbf{n} = \frac{2}{3}\,q^e\mathbf{n} - 2G\varepsilon_q\mathbf{n} \quad \rightarrow \quad q = q^e - 3G\varepsilon_q$$

For later we'll need the derivatives:

$$\frac{\partial p}{\partial\varepsilon_p} = 3K \quad ; \quad \frac{\partial q}{\partial\varepsilon_q} = -3G$$

## 2. Rousselier's plastic potential

Rousselier's notation uses $\sigma_m$ and $\sigma_{eq}$ which in our notation are $p = -\sigma_m$ and $q = \sigma_{eq}$. The hardening (flow) function is $H(\varepsilon_q)$. In that notation the Rousselier's plastic potential is:

$$g = \frac{q}{\rho} + BD \exp \frac{-p}{\rho\sigma_1} - H$$

where all other terms are as in (Rousselier, 1987).

So the derivatives are:

$$\frac{\partial g}{\partial p} = \frac{-1}{\rho\sigma_1} BD \exp \frac{-p}{\rho\sigma_1} \quad ; \quad \frac{\partial g}{\partial q} = \frac{1}{\rho}$$

with that the normality rule becomes:

$$\varepsilon_q \frac{-1}{\rho\sigma_1} BD \exp \frac{-p}{\rho\sigma_1} + 3\varepsilon_p \frac{1}{\rho} = 0$$

or

$$\varepsilon_p - \varepsilon_q \frac{BD}{3\sigma_1} \exp \frac{-p}{\rho\sigma_1} = 0$$

which is identical to Eqn. (38)b in (Rousselier, 1987).

Finally, $H$ is the hardening function of the total equivalent plastic strain, $\mathrm{E}_q$:

$$H = H(\mathrm{E}_q)$$

where $\mathrm{E}_q = \mathrm{E}_q(t) + \varepsilon_q$. Hence

$$H^{'} = \frac{dH}{d\varepsilon_q} = \frac{\partial H}{\partial \mathrm{E}_q} \frac{d\mathrm{E}_q}{d\varepsilon_q} = \frac{\partial H}{\partial \mathrm{E}_q}$$

## 3. Damage variable and density

The system of equations becomes complete with the addition of the damage evolution function and the expression for density. These are taken directly from Eqns. (39)b, (45) and (46) in (Rousselier, 1987). In our notation these are:

$$\Delta\beta = \varepsilon_q D \exp \frac{-p}{\rho\sigma_1}$$

$$\rho(\Delta\beta) = (1 - f_0 + f_0 \exp \beta)^{-1}$$

$$B(\Delta\beta) = \sigma_1 f_0 \exp \beta \rho$$

where $\beta = \beta_t + \Delta\beta$, and $\beta_t$ is the damage variable saved from the previous time/load increment.

For later we'll need the derivatives:

$$\frac{\partial\rho}{\partial\Delta\beta} = -(1 - f_0 + f_0 \exp \beta)^{-2} f_0 \exp \beta = -\rho^2 f_0 \exp \beta$$

$$\frac{\partial B}{\partial\Delta\beta} = \sigma_1 f_0 \left( \frac{\partial \exp \beta}{\partial\Delta\beta} \rho + \exp \beta \frac{\partial\rho}{\partial\Delta\beta} \right) = \sigma_1 f_0 (\exp \beta \rho - \rho^2 f_0 (\exp \beta)^2) = \sigma_1 f_0 \rho \exp \beta (1 - \rho f_0 \exp \beta)$$

## 4. Numerical problem formulation

There are 3 unknowns: $\varepsilon_p$, $\varepsilon_q$ and $\Delta\beta$. These are found by a solution of a system of 3 non-linear equations - plastic potential equal to zero, normality rule and the evolution of the damage variable. Using optimisation notation we can formulate the problem as:

$$\mathbf{x} = (\varepsilon_p, \varepsilon_q, \Delta\beta)$$

$$\mathbf{f} = \mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}),\ f_2(\mathbf{x}),\ f_3(\mathbf{x}))$$

We want to minimise some norm of **f**, most typically the 2nd norm. We denote by $\mathbf{x}^*$ the vector of arguments that minimises the norm.

$$\mathbf{x}^* = \min_{\mathbf{x}} |\mathbf{f}(\mathbf{x})|_2$$

The numerical solution will require the Jacobian:

$$\mathbf{J} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}$$

In the following we derive explicit expressions for **f** and **J**.

$$f_1 = g = \frac{q}{\rho} + BD \exp \frac{-p}{\rho\sigma_1} - H = 0$$

$$f_2 = \varepsilon_p - \varepsilon_q \frac{BD}{3\sigma_1} \exp \frac{-p}{\rho\sigma_1} = 0$$

$$f_3 = \Delta\beta - \varepsilon_q D \exp \frac{-p}{\rho\sigma_1} = 0$$

or introducing

$$z = z(\varepsilon_p, \Delta\beta) = D \exp \frac{-p}{\rho\sigma_1}$$

$$\frac{\partial z}{\partial \varepsilon_p} = z \frac{-1}{\rho\sigma_1} \frac{\partial p}{\partial \varepsilon_p} \quad ; \quad \frac{\partial z}{\partial \Delta\beta} = z \frac{-p}{\sigma_1} \frac{-1}{\rho^2} \frac{\partial \rho}{\partial \Delta\beta} = z \frac{p}{\sigma_1 \rho^2} \frac{\partial \rho}{\partial \Delta\beta}$$

the system can be written as:

$$f_1 = g = \frac{q}{\rho} + Bz - H = 0$$

$$f_2 = \varepsilon_p - \varepsilon_q \frac{Bz}{3\sigma_1} = 0$$

$$f_3 = \Delta\beta - \varepsilon_q z = 0$$

Components of Jacobian:

$$\frac{\partial f_1}{\partial x_1} = \frac{\partial f_1}{\partial \varepsilon_p} = B \frac{\partial z}{\partial \varepsilon_p} \quad ; \quad \frac{\partial f_1}{\partial x_2} = \frac{\partial f_1}{\partial \varepsilon_q} = \frac{1}{\rho} \frac{\partial q}{\partial \varepsilon_q} - H' \quad ; \quad \frac{\partial f_1}{\partial x_3} = \frac{\partial f_1}{\partial \Delta\beta} = q \frac{-1}{\rho^2} \frac{\partial \rho}{\partial \Delta\beta} + B \frac{\partial z}{\partial \Delta\beta} + \frac{\partial B}{\partial \Delta\beta} z$$

$$\frac{\partial f_2}{\partial x_1} = \frac{\partial f_2}{\partial \varepsilon_p} = 1 - \frac{\varepsilon_q B}{3\sigma_1} \frac{\partial z}{\partial \varepsilon_p} \quad ; \quad \frac{\partial f_2}{\partial x_2} = \frac{\partial f_2}{\partial \varepsilon_q} = -\frac{Bz}{3\sigma_1} \quad ; \quad \frac{\partial f_2}{\partial x_3} = \frac{\partial f_2}{\partial \Delta\beta} = -\frac{\varepsilon_q}{3\sigma_1} \left( \frac{\partial B}{\partial \Delta\beta} z + B \frac{\partial z}{\partial \Delta\beta} \right)$$

$$\frac{\partial f_3}{\partial x_1} = \frac{\partial f_3}{\partial \varepsilon_p} = -\varepsilon_q \frac{\partial z}{\partial \varepsilon_p} \quad ; \quad \frac{\partial f_3}{\partial x_2} = \frac{\partial f_3}{\partial \varepsilon_q} = -z \quad ; \quad \frac{\partial f_3}{\partial x_3} = \frac{\partial f_3}{\partial \Delta\beta} = 1 - \varepsilon_q \frac{\partial z}{\partial \Delta\beta}$$

## References

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